SOME NEW TECHNIQUES IN THE
DYNAMIC PROGRAMMING SOLUTION
OF VARIATIONAL PROBLEMS

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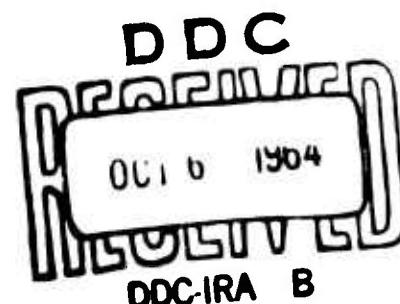
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SUMMARY

In previous papers, it has been shown that the functional
equation technique of dynamic programming may be applied to
yield the numerical solution of a wide class of variational
problems of the type occurring in mathematical physics,
engineering, and economics.

It was seen that the numerical solution of a problem
involving N state variables depended upon the computation
of sequences of functions of N variables. This fact made
the method routine only for the case where N = 1 or 2, with
grave difficulties arising in the general case.

In this paper, it is indicated how to overcome this
difficulty for a large class of problems in which the underlying
equations and the criterion function are linear, although the
restraints on the forcing functions may be nonlinear, corres-
ponding say to energy considerations.

The same methods are applicable to other classes of linear
equations, and, in particular, to differential-difference
equations, arising from time-lag problems, and to various
classes of partial differential equations. These problems
could not previously be treated by dynamic programming tech-
niques in any useable fashion.

Finally, it is briefly indicated how the method of
successive approximations may be combined with the foregoing
techniques to reduce general variational problems, in which the

equations and criterion function are nonlinear, to sequences
of problems which can be solved numerically by means of
sequences of functions of one variable. There are a number of
interesting and difficult convergence questions associated
with this program which we do not discuss here.

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§1.   INTRODUCTION

A variational problem that is encountered in many parts of pure and applied mathematics is that of determining the minimum or maximum of a functional of the form

$$(1) \qquad J(y) = \int_0^T F(x_1, x_2, \ldots, x_N; y_1, y_2, \ldots, y_M)dt$$

over all functions $y_1, y_2, \ldots, y_M$ satisfying the relations

$$(2) \qquad \frac{dx_1}{dt} = G_1(x;y), \quad x_1(0) = c_1, \quad i = 1, 2, \ldots, N,$$

and constraints of the form

$$(a) \quad x_1(T) = b_1, \quad i = 1, 2, \ldots, R,$$
$$(3)$$
$$(b) \quad R_j(x;y) \leq 0, \quad j = 1, 2, \ldots, k.$$

For a variety of reasons which one discovers almost immediately, this problem, to the degree of generality stated above, presents formidable difficulties. These difficulties arise not only in connection with the analytic solution of the problem, cf. [9], [10], but also in connection with the apparently more modest demand for a numerical solution. We have discussed elsewhere various applications of the theory of dynamic programming to the numerical solution of classes of

variational problems of the foregoing type, [1], [2], [3], [4].
In this paper we wish to indicate some recent developments
which greatly enlarge the scope of these methods presented in
the cited references. In addition, these new developments,
combined with the classical tool of successive approximations,
enable us to attack systematically classes of problems formerly
far beyond our powers.

We shall begin our discussion with a terminal control
problem for a linear system with constant coefficients. The
problem is that of maximizing a functional of the form

$$(4) \qquad J(y) = H(x_1(T), x_2(T), \ldots, x_k(T))$$

over all functions $y_1, y_2, \ldots, y_M$, where x and y are
linked by linear relations of the form

$$(5) \qquad \frac{dx_1}{dt} = \sum_{j=1}^{N} a_{1j}x_j + \sum_{j=1}^{N} b_{1j}y_j, \quad x_1(0) = c_1, \quad i = 1, 2, \ldots, N$$

and the y_1 satisfy restraints of the type

$$(a) \quad m_1 \leq y_1(t) \leq m_1', \quad 0 \leq t \leq T, \quad i = 1, 2, \ldots, N,$$

$$(6)$$

$$(b) \quad \int_0^T K_j(y_1, y_2, \ldots, y_N)dt \leq c_j, \quad j = 1, 2, \ldots, \ell.$$

Whereas the techniques of dynamic programming discussed
previously convert this variational problem into one requiring
the computation of a sequence of functions of N variables,
the linearity of the defining equation in (5) enables us to

transform the problem into one involving a sequence of functions of k variables, where k is as in (4).

Since current digital computers do not take kindly to storage of functions of more than two variables, this is a very important reduction. This is particularly so since a large class of problems occurring in the fields of engineering and industrial economics may be formulated in the above terms with large N and a k of one or two.

If $H(x_1, x_2, \ldots, x_k)$ is linear,

$$(7) \qquad H = \sum_{j=1}^{k} \alpha_j x_j,$$

the problem may be still further reduced to the computation of sequences of functions of one variable, regardless of the value of k. The same remark holds for the case where it is desired to maximize a linear functional of the form

$$(8) \qquad J = \int_0^T \left\{ \sum_{j=1}^{N} \beta_j x_j(t) \right\} dt.$$

In exactly the same way we can handle discrete control problems where difference equations replace differential equations. As a matter of fact, these techniques were developed in connection with a discrete problem, the "caterer" problem, [5].

The same methods may be applied to the treatment of problems involving time lags and retarded control, cf. [7], in which case the equations corresponding to (5) become

$$\frac{dx_1}{dt} = \sum_{j=1}^{N} a_{ij}x_j(t) + \sum_{j=1}^{N} \alpha_{ij}x_j(t-\delta) + \sum_{j=1}^{N} b_{ij}y_j(t),$$

(9)

$$y_i(t) = c_i(t), \quad 0 \le t \le \delta, \quad i = 1, 2, \ldots, N.$$

More complicated types of hereditary processes may also be treated by means of the techniques we shall present below. Problems of this kind could not formerly be treated by dynamic programming techniques because of their dependence upon functionals rather than functions.

The results stated so briefly in the foregoing paragraphs depend in an essential manner upon the linearity of the equations defining the process and upon the linearity of the criterion function. To extend these techniques to cover more general situations, we turn to that general factotum, the method of successive approximations. Using this basic technique of analysis, we show how a variety of apparently multi-dimensional processes can be reduced to computational processes involving sequences of functions of one variable. Once this has been done we have a feasible approach to these problems.

A large number of interesting and significant problems involving convergence of these methods, rapidity of convergence, stability, and so on, arise from these investigations. These will be discussed in detail at a subsequent date.

§2. DYNAMIC PROGRAMMING

In order to appreciate the improvement in technique

afforded by the methods we present here, let us sketch briefly the direct approach of dynamic programming which we have presented in the works cited above.

Considering the general problem posed in (1.1)–(1.3) define the function of the N variables $c_1, c_2, \ldots, c_N$, and T,

$$(1) \qquad f(c_1, c_2, \ldots, c_N; T) = \operatorname*{Max}_{y} J(y)$$

where the maximum is taken over all functions $y_i(t)$ satisfying (1.3). Under appropriate assumptions concerning the continuous dependence of maximizing y_i upon the initial values c_i and T, we obtain for f a nonlinear partial differential equation

$$(2) \qquad \frac{\partial f}{\partial T} = \operatorname*{Max}_{v_i} \left[F(c;v) + \sum_{i=1}^{N} G_i(c,v) \frac{\partial f}{\partial c_i} \right], \quad f(c_1, c_2, \ldots, c_N, 0) = 0,$$

where the maximization is over quantities v_i satisfying the constraints

$$(a) \quad m_i \leq v_i \leq m_i'$$
$$(3)$$
$$(b) \quad R_j(c,v) \leq 0.$$

Provided that F and G are suitably differentiable, and assuming that the maximum is always assumed inside the region of variation, this nonlinear partial differential equation leads via characteristics to the usual Euler equations, cf. [1], [12].

§3. COMPUTATIONAL ASPECTS

Since the analytic solution of these problems, as mentioned above, is only rarely attained, we turn to computational techniques.

To determine $f(c_1, c_2, \ldots, c_N, T)$ we can either use (2.2) and any of a number of standard techniques for the numerical solution of partial differential equations of this type, or, as has turned out to be preferable, we can go over to a discrete version of the original continuous process. Analytically, this means that the original differential equations are replaced by difference equations. Thus, (1.2) becomes

$$(1) \qquad x_1(t+\delta) = x_1(t) + \delta \ G_1(x_1(t), x_2(t), \ldots, x_N(t);$$

$$y_1(t), \ldots, y_M(t)), \ x_1(0) = c_1,$$

with t assuming only the values $0, \delta, 2\delta, \ldots$.

The nonlinear partial differential equation is then replaced by the nonlinear recurrence relation

$$f(c, T) = \underset{v}{\text{Max}} \ [\delta \ F(c; v) + f(c_1 + \delta G_1(c, v), \ldots, c_N$$

$$(2)$$

$$+ \delta G_N(c, v); \ T - \delta)] \ .$$

In order to carry out the indicated process, we must be able to tabulate functions of N variables. Consequently, at the present time this approach is only feasible if $N = 1$

or 2. For $N > 2$, the memory requirements become prohibitive.

It is necessary then to develop some new techniques if we wish to utilize dynamic programming to solve large scale problems of the type arising in the engineering and economic spheres.

§4. PRELIMINARIES ON LINEAR SYSTEMS

In this section we shall mention some well-known results concerning the solution of vector-matrix systems of linear differential equations. These will be utilized in what follows. Proofs of the results cited here may be found in [6].

The linear system of (1.5) may be written, using an obvious vector-matrix notation, in the form

(1) $$\frac{dx}{dt} = Ax + By, \quad x(0) = c.$$

Consider first the case in which A is constant. The solution of (1) may then be written in the form

(2) $$x = e^{At}c + \int_0^t e^{A(t-s)}By(s)ds.$$

If $A = A(t)$, a matrix dependent upon t, then x may be written in the form

(3) $$x = X(t)c + \int_0^t X(t)X^{-1}(s)By(s)ds,$$

where $X(t)$ is the matrix solution of

(4) $$\frac{dX}{dt} = A(t)X, \quad X(0) = I.$$

We shall utilize these representations in a crucial

manner below.

5.   TERMINAL CONTROL—I

Let us now turn to the problem of maximizing a given

function $H(x_1(T), x_2(T), \ldots, x_k(T))$ of the terminal state of

the system over all control functions $y_1(t)$ which are re-

lated to the x_1 by means of the linear equation in (4.1),

and which are subject to the constraints

(a) $m_1 \leq y_1(t) \leq m_1'$, $0 \leq t \leq T$, $1 = 1, 2, \ldots, N$,

(1)

(b) $\int_0^T G(y_1, y_2, \ldots, y_N) dt \leq k$.

We wish to show that the numerical solution of a problem

of this type can be made to depend upon a sequence of

functions of k variables, rather than upon sequences of

functions of N variables. We shall consider first the case

where $A = (a_{1j})$ is a constant matrix.

We begin with the linear representation of (4.2), which

yields a set of equations

$$x_1(t) = z_1(t) + \int_0^t \left[\sum_{j=1}^{N} x_{1j}(t-s) y_j(s) \right] ds,$$

(2)

$$1 = 1, 2, \ldots, N,$$

where $z_1(t)$ is the 1-th component of $e^{At} c$, and $X(t) = (x_{1j}(t))$.

The problem we wish to consider may then be cast in the

form of maximizing a functional of the type

$$H(u_1 + \int_0^T [\sum_{j=1}^N x_{1j}(T-s)y_j(s)]ds, \ldots, u_k$$

(3)

$$+ \int_0^T [\sum_{j=1}^N x_{kj}(T-s)y_j(s)]ds),$$

where u_1, u_2, $\ldots$, u_k are given quantities, over all functions y_1, y_2, $\ldots$, y_N satisfying the constraints (1a) and (1b).

Let us then consider the sequence of functions $f(u_1, u_2, \ldots, u_k; T)$, implicitly dependent upon λ, defined as follows

$$f(u_1, u_2, \ldots, u_k; T) = \underset{y}{\text{Max}} [H(u_1 + \int_0^T [\ldots]ds, \ldots, u_k$$

(4)

$$+ \int_0^T [\ldots]ds) - \lambda \int_0^T G(y_1, y_2, \ldots, y_N)ds],$$

where the functions $y_i(t)$ are now constrained by (1a).

A motivation and discussion of the use of the Lagrange multiplier may be found in [9], and a numerical example in [11]. The value of the method lies in the fact that enables us to reduce multi-dimensional problems to sequences of lower dimensional problems.

To obtain a functional equation for $f(u_1, u_2, \ldots u_k; T)$, we proceed as follows. Suppose that the values of $y_1(t)$, $y_2(t)$, $\ldots$, $y_N(t)$ have been determined over $[0, \delta]$. Then we may write

$$H(u_1 + \int_0^T [\ldots]ds, \ldots, u_k + \int_0^T [\ldots]ds)$$

$$- \lambda \int_0^T G(y_1, y_2, \ldots, y_N)ds$$

$$= H(u_1 + \int_0^\delta [\ldots]ds + \int_\delta^T [\ldots]ds, \ldots, u_k$$

$$+ \int_0^\delta [\ldots]ds + \int_\delta^T [\ldots]ds)$$

(5)
$$- \lambda \int_0^\delta G(y_1, y_2, \ldots, y_N)ds - \lambda \int_\delta^T G(y_1, y_2, \ldots, y_N)ds$$

$$= H(u_1 + \int_0^\delta [\ldots]ds$$

$$+ \int_0^{T-\delta} [\sum_{j=1}^N x_{1j}(T-\delta-s)y_j(s+\delta)]ds, \ldots)$$

$$- \lambda \int_0^\delta G(y_1, y_2, \ldots, y_N)ds$$

$$- \lambda \int_0^{T-\delta} G(y_1(s+\delta), \ldots, y_N(s+\delta))ds.$$

The principle of optimality cf. [1] then yields the
functional equation

$$f(u_1, u_2, \ldots, u_k; T) = \underset{y[0,\delta]}{\text{Max}} [- \lambda \int_0^\delta G(y_1, y_2, \ldots, y_N)ds +$$

(6)
$$f(u_1 + \int_0^\delta [\sum_{j=1}^N x_{1j}(T-s)y_j(s)]ds, \ldots, u_k$$

$$+ \int_0^\delta [\sum_{j=1}^N x_{kj}(T-s)y_j(s)]ds)].$$

The maximum is now taken over all functions $y_1(s)$ defined over $0 \leq s \leq \delta$, and satisfying the constraints $m_1' \leq y_1(s) \leq m_1$ in $[0,\delta]$.

For computational purposes, we may use the approximate relation

$$
f(u_1, u_2, \ldots, u_k; T) = \operatorname*{Max}_{v} \left[- \lambda\delta \, G(v_1, v_2, \ldots, v_N) + \right.
$$

(7)
$$
f(u_1 + \delta \sum_{j=1}^{N} x_{1j}(T)v_j, \ldots, u_k +
$$

$$
\left. \delta \sum_{j=1}^{N} x_{kj}(T)v_j) \right],
$$

or we may start with a discrete version of the original process.

We have thus reduced the numerical solution of the variational problem to the determination of a sequence of functions of k variables. If $k = 1$ or 2, we have a feasible method of solution.

9. TERMINAL CONTROL—VARIABLE COEFFICIENTS

It is important in connection with our subsequent discussion of the use of successive approximations to consider the same problem for the case where A is a variable matrix. Let us consider then the case where the equation governing the process has the form

(1)
$$
\frac{dx}{dt} = A(t)x + B(t)y + \phi(t), \quad x(0) = c.
$$

As we know, the solution of this equation is given by the expression

$$x = X(t)c + \int_0^t X(t)X^{-1}(s)B(s)y(s)ds +$$

(2)

$$\int_0^t X(t)X^{-1}(s)\phi(s)ds.$$

Hence the components of $x(T)$ has the form

(3)
$$x_1(T) = u_1 + \int_0^T [\sum_{j=1}^N x_{1j}(T,s)y_j(s)]ds,$$

where the u_1 are independent of y.

In order to take account of the non—stationarity of the process, we count time backwards. In place of noting the time at which the process ends, we single out the time at which it begins. Fixing T, we consider the function $f(u_1,u_2,\ldots,u_k;r)$ defined by the relation

$$f(u_1,u_2,\ldots,u_k;r) = \underset{y[r,T]}{\text{Max}} \ [H(u_1 +$$

(4)
$$\int_r^T [\sum_{j=1}^N w_{1j}(T,s)y_j(s)]ds, \ldots, u_k +$$

$$\int_r^T [\ldots]ds) - \lambda \int_r^T G(y_1,y_2,\ldots,y_N)ds].$$

Arguing as in the preceding section, we see that f satisfies the relation

$$f(u_1,u_2,\ldots,u_k;r) = \underset{y[r,r+\delta]}{\text{Max}} \ [- \lambda \int_r^{r+\delta} G(y_1,y_2,\ldots,y_N)ds$$

(5)
$$+ f(u_1 + \int_r^{r+\delta} [\ldots]ds,\ldots,u_k$$

$$+ \int_r^{r+\delta} [\ldots] ds)].$$

For computational purposes, this reduces to

$$f(u_1, u_2, \ldots, u_k; r) = \underset{v}{\text{Max}} \; [- \lambda \delta \; G(v_1, v_2, \ldots, v_N) +$$

(6)

$$f(u_1 + \delta \sum_{j=1}^{N} w_{1j}(T,r)v_j, \; \ldots, \; u_k +$$

$$\delta \sum_{j=1}^{N} w_{kj}(T,r)v_j)],$$

with $f(u_1, u_2, \ldots, u_k; T) \equiv 0.$

§7. TERMINAL CONTROL—LINEAR CRITERION

Let us now consider the case where H is a linear function. More generally, let us consider the problem of maximizing the inner product $(x(T), a)$ where a is a given vector. To simplify the notation, let us consider only the case where A is a constant matrix.

Using the representation for $x(t)$ given in (4.2), we see that

(1) $\qquad (x(T), a) = (e^{AT}c, a) + (\int_0^T e^{A(T-s)}y(s)ds, a).$

Neglecting the term $(e^{AT}c, a)$, which is independent of y, we have the problem of maximizing

(2) $\qquad J(y) = (\int_0^T e^{A(T-s)}y(s)ds, a)$

over all y_1 satisfying the constraints

(3)

(a) $m_1 \leq y_1 \leq m_1'$, $0 \leq t \leq T$, $1 = 1, 2, \ldots, N$,

(b) $\int_0^T G(y_1, y_2, \ldots, y_N)ds \leq k.$

Introduce the function

(4) $$f(k,T) = \underset{y}{\text{Max}}\ J(y),$$

where the maximum is over all functions y_1 satisfying (3a) and (3b).

It is easy to see that $f(k,T)$ satisfies the equation

(5)

$$f(k,T) = \underset{y[0,\delta]}{\text{Max}}\ [(\int_0^\delta e^{A(T-s)}yds, a)$$

$$+ f(k - \int_0^\delta G(y_1, y_2, \ldots, y_N)ds, T-\delta)],$$

$$f(k,0) = 0.$$

For computational purposes, we can use the relation

(6) $$f(k,T) = \underset{v}{\text{Max}}\ [\delta(e^{AT}v, a) + f(k-\delta\ G(v_1, v_2, \ldots, v_N)T-\delta)].$$

We see then that in the case where the underlying equation is linear, the criterion function is linear, and there is only one constraint of the form in (3b), we can compute the solution using sequences of functions of one variable.

If there are two constraints of the type

(7) $\qquad \int_0^T G_1(y_1, y_2, \ldots, y_N) ds \leq k_1, \quad i = 1, 2,$

we introduce a Lagrange multiplier and consider the problem of maximizing

(8) $\qquad (x(T), a) - \lambda \int_0^T G_1(y_1, y_2, \ldots y_N) ds.$

For each value of λ we have a one—dimensional problem. As the parameter λ is varied, we range over a set of values of the constraint $\int_0^T G_1(y_1, y_2, \ldots, y_N) ds.$

§8. TERMINAL CONTROL VS. GENERAL CONTROL

The general control problem is that of maximizing a functional of the form

(1) $\qquad J(y) = \int_0^T F(x, y) dt + H(x(T)),$

subject to relations and constraints of the type described above.

If we introduce a new scalar variable $x_{N+1}(t)$ by means of the relation

(2) $\qquad \dfrac{dx_{N+1}}{dt} = F(x, y), \quad x_{N+1}(0) = 0,$

we see that we once again have a terminal control problem.

In particular, the problem of maximizing a linear functional of the form

(3) $$\int_0^T [\sum_{j=1}^N [\alpha_j x_j(t) + \beta_j y_j(t)]]dt,$$

is readily treated by the above methods.

$9. TIME LAG PROBLEMS

As we have seen in the foregoing sections, the success of our methods rested upon the superposition principle. Given an equation of the form

(1)
$$L(x) = y,$$

$$x_{t=0} = c,$$

we were able to write the solution in the form

(2) $$x = \dot{X}(t)c + T(y),$$

and thus avoid interaction between the initial conditions and the forcing function.

Put in these terms, it is clear that the same methods will handle the case where the linear operator L is represented by a differential—difference operator of the type given in (1.9). The requisite representation theorems may be found in [7].

$10. PARTIAL DIFFERENTIAL EQUATIONS

Similarly, a variety of control problems associated with the heat equation

$$u_t = u_{xx} + v(x,t),$$

(1) $u(x,0) = c(x), \quad 0 \le x \le 1$

$$u(0,t) = u(1,t) = 0, \quad t > 0,$$

may be treated.

We shall reserve a detailed exposition of problems of this type for a future date.

§11. SUCCESSIVE APPROXIMATIONS

Let us now briefly, without entering into any rigorous discussion, which is non—trivial, indicate how the method of successive approximations may be combined with the foregoing techniques so as to reduce general control problems involving nonlinear differential equations and nonlinear criteria to sequences of computational problems involving functions of one variable.

Consider the problem of maximizing

(1) $J(y) = H(x_1(T), x_2(T), \ldots, x_k(T))$

over all y subject to

(a) $\dfrac{dx_1}{dt} = G_1(x,y), \quad x_1(0) = c_1, \quad i = 1, 2, \ldots, N,$

(2) (b) $m_1 \le y_1 \le n_1', \quad i = 1, 2, \ldots, N,$

(c) $\displaystyle\int_0^T G(y_1, y_2, \ldots, y_N)\,dt \le k.$

Let $y^{(0)} = (y_1^{(0)}, y_2^{(0)}, \ldots, y_N^{(0)})$ be an initial guess in policy space, and let $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_N^{(0)})$ be the resultant x-values, determined by (2a).

Consider the new criterion function

$$(3) \quad J_0(y) = H(x_1^{(0)}(T), x_2^{(0)}(T), \ldots, x_k^{(0)}(T)) +$$
$$\sum_{i=1}^{k} [x_i(T) - x_i^{(0)}(T)] \frac{\partial H}{\partial x_i} (x_1^{(0)}(T), \ldots, x_k^{(0)}(T)),$$

and the new system of differential equations

$$(4) \quad \frac{dx_i}{dt} = G_i(x^0, y) + \sum_{j=1}^{N} (x_j - x_j^{(0)}) \frac{\partial f_i}{\partial x_j} (x^{(0)}, y),$$

$x_i(0) = c_i, \quad i = 1, 2, \ldots, N.$

The new variational problem is that of maximizing $J_0(y)$ over all y satisfying (4) and the constraints (2b) and (2c).

Since the criterion function and the system of differential equations are linear, the methods outlined in the foregoing pages permit us to solve this problem by means of a sequence of one-dimensional functions.

Let the maximizing $y(t)$ be called $y^{(1)}(t)$. Proceeding as above, we determine an $x^{(1)}(t)$, and formulate a new variational problem as in (3) and (4). Proceeding in this way we form a sequence of one-dimensional problems which we hope converges to the solution of the original problem.

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